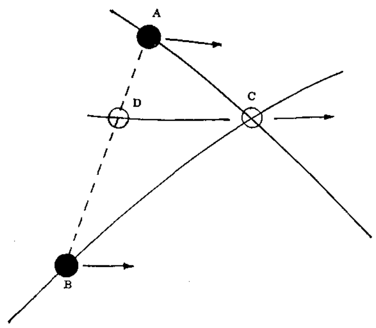

Fig. 2 Field point geometry

The dominant feature of any hypersonic flow field is the appearance of very strong gradients in the flow properties. However, as just noted, the usual numerical method-of-characteristics program relies on an assumption of constant entropy gradient between nearby field points. It is this assumption that introduces the error into the flow field calculations.

The remedy is quite simple. For bodies without secondary shocks, the relation between the value of the mass integral and entropy is established at the shock wave and is invariant (the value of the mass integral assigned to a particular streamline is defined as the amount of mass flowing between the streamline and the body, i.e., a Stokes stream-function). For bodies with secondary shocks, the relation is not invariant, but the philosophy of calculation is the same; the influence of the imbedded shock can be accounted for easily. In order to calculate a new field point properly, one may use this mass-entropy curve in the following manner.

From the data given at points A and B and the compatibility relations in finite difference form, the pressure and flow direction at C are calculated as before. A first guess as to the entropy at C is made by, say, averaging the entropy at A and B. This permits one to determine the Mach number at point C. The mass integral value of the streamline through A is known, and let it be mass_A. The initial value of the mass integral at the shock is, of course,

$$\text{mass}_{\text{shock}} = (2\pi y_{\text{shock}})^{\epsilon} \rho_\infty V_\infty y_{\text{shock}}$$

The appropriate value of the mass flow at C is then given by

$$\text{mass}_C = \text{mass}_A = (2\pi y)^{\epsilon} <\rho a y> \Delta s$$

where $<\rho a y>$ is the average of the product of density, speed of sound and ordinate at points A and C, $\Delta s = (\Delta x^2 + \Delta y^2)^{1/2}$

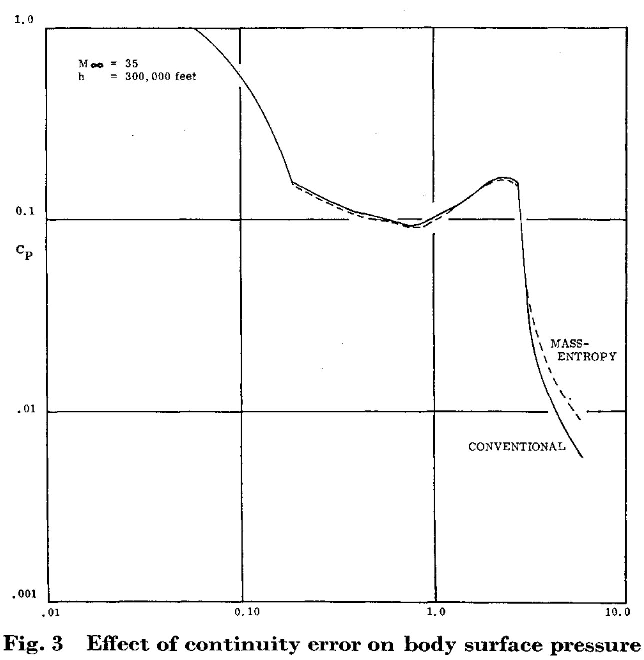

Fig. 3 Effect of continuity error on body surface pressure

and $\epsilon = 0$ or 1 for two-dimensional or axisymmetric flow, respectively. Using the value of mass, and the universal mass-entropy relation established at the shock wave, a new estimate for the entropy can be made. By iterating this process until no further changes occur, a good estimate of the local entropy can be established. The next overall cycle of iteration using the compatibility relations then can be carried out, and the entire process repeated until convergence occurs.

The mass flow ratio using this "mass-entropy" method is given in Fig. 1 where identical mesh sizes were used for both cases. Note that a large reduction in the continuity error has been achieved. It is evident that this new method is superior to the older method of assuming a linear variation of flow properties between adjacent field points.

The influence of the error shown in Fig. 1 on the body pressure distribution is small, as shown in Fig. 3. Only near the base of the body does any appreciable difference occur. The principal damage done by the mass error is in the flow field itself where in some cases an order-of-magnitude difference between quantities calculated by the "mass-entropy" method and the usual practice has been found.

The method of characteristics program used to generate the data presented in this note now has been extended to handle secondary and envelope shocks by the mass-entropy method with the same degree of overall improvement. The mass error value also has proven to be an invaluable guide in uncovering errors in the calculation process.

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Gyroscopic Stabilization of Space Vehicles

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The gyroscopic stabilization of a space vehicle containing a simple disk gyro is investigated. The governing dynamical equations are derived and applied to two specific cases. The first case is a study of the stability of uniform rotation with uniform gyro speed. The second case considers stability of uniform rotation with an oscillating gyro. The results indicate that the stability is highly dependent upon the motion of the gyro and that stabilization may be attained independently of the inertia properties of the vehicle.

RECENTLY there has been an increasing interest in inertia or gyroscopic stabilization. This is especially true in the field of the mechanics of space vehicles where it is advantageous to attain stability independently of external forces or reaction jets. In connection with this, it is the purpose of this paper to investigate such stabilization by seeking stability criteria for a particular space vehicle-gyro system.†

Received by ARS December 12, 1962.

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† The author initially received the idea for this investigation in a course given by T. R. Kane at the University of Pennsylvania.

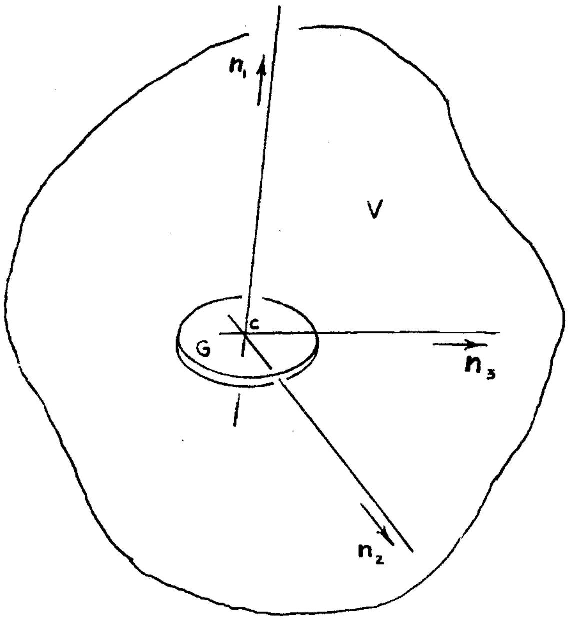

Fig. 1 Vehicle with gyro

Governing Equations

The system to be considered is depicted in Fig. 1 where V is a rigid vehicle, which has mounted within it a circular disk gyro G whose mass center c coincides with the mass center of V. The axis of G is fixed in V and furthermore, this axis is parallel to one of the principal inertia axes of V with respect to c. Finally, in Fig. 1, $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ represent a right-handed set of mutually perpendicular unit vectors that remain parallel to the three principal inertia axes of V with respect to c.

Consider V situated in a space such that the only externally applied forces are gravitational and that these forces may be replaced by a single force whose line of action passes through c. Equations describing the motion of the system may be obtained from the following vector equation that arises when moments of all forces are taken about c:

$$\mathbf{T}_V + \mathbf{T}_G = 0 \tag{1}$$

$\mathbf{T}_V$ and $\mathbf{T}_G$ are the respective "inertia torques" on V and G and they are computed in a Newtonian reference frame. Expressing these torques in terms of the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$, one obtains measure numbers of the type[1]

$$T_{V1} = -(d\omega_1/dt)I_1 + \omega_2\omega_3(I_2 - I_3) \tag{2}$$

and

$$T_{G1} = -\alpha_{G1}I_{G1} + \omega_{G2}\omega_{G3}(I_{G2} - I_{G3}) \tag{3}$$

where ω_i and ω_{Gi} ($i = 1,2,3$) represent the respective measure numbers of the angular velocities of V and G, I_i and I_{Gi} represent the principal moments of inertia of V and G with respect to c, and α_{Gi} represents the measure numbers of the angular acceleration of G. The subscripts in these symbols refer to the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$.

By recognizing the symmetry properties of G, its inertia properties may be expressed more conveniently through the relation

$$\tfrac{1}{2}I_{G1} = I_{G2} = I_{G3} \equiv I \tag{4}$$

Furthermore if G rotates about its axis with an angular speed Ω measured in V, its angular velocity in a Newtonian reference frame may be expressed as[2]

$$\boldsymbol{\omega}_G = \Omega\mathbf{n}_1 + \boldsymbol{\omega}_V \tag{5}$$

where $\boldsymbol{\omega}_V$ represents the angular velocity of V in the Newtonian reference frame. Hence, $\boldsymbol{\omega}_G$ may be expressed in the form

$$\boldsymbol{\omega}_G = (\Omega + \omega_1)\mathbf{n}_1 + \omega_2\mathbf{n}_2 + \omega_3\mathbf{n}_3 \tag{6}$$

The angular acceleration of G in the Newtonian frame $\boldsymbol{\alpha}_G$ may be obtained by taking the time derivative of $\boldsymbol{\omega}_G$. Noting that the $\mathbf{n}_i$ are fixed in V, $\boldsymbol{\alpha}_G$ is found to be

$$\boldsymbol{\alpha}_G = \left(\frac{d\Omega}{dt} + \frac{d\omega_1}{dt}\right)\mathbf{n}_1 + \left(\frac{d\omega_2}{dt} + \omega_3\Omega\right)\mathbf{n}_2 + \left(\frac{d\omega_3}{dt} - \omega_2\Omega\right)\mathbf{n}_3 \tag{7}$$

By identifying ω_{Gi} and α_{Gi} in Eqs. (6) and (7) and using Eqs. (2, 3, and 4), the following equations of motion then are obtained from Eq. (1):

$$(I_1 + 2I)(d\omega_1/dt) + (I_3 - I_2)\omega_2\omega_3 + 2I(d\Omega/dt) = 0 \tag{8}$$

$$(I_2 + I)(d\omega_2/dt) + (I_1 - I_3)\omega_3\omega_1 + 2I\Omega\omega_3 + I\omega_1\omega_3 = 0 \tag{9}$$

$$(I_3 + I)(d\omega_3/dt) + (I_2 - I_1)\omega_1\omega_2 - 2I\Omega\omega_2 - I\omega_1\omega_2 = 0 \tag{10}$$

Case I: Stability of uniform rotation with uniform gyro speed

Consider the case when G is rotating in V with constant speed while V is rotating about an axis that is parallel to $\mathbf{n}_1$.

In order to investigate the stability of this motion a small disturbance is introduced and conditions are sought for which this disturbance remains small. Specifically, the angular velocity of V is given the form

$$\boldsymbol{\omega}_V = (\omega_0 + \omega_1^*)\mathbf{n}_1 + \omega_2^*\mathbf{n}_2 + \omega_3^*\mathbf{n}_3 \tag{11}$$

where the starred terms are small and ω_0 is a constant. By identifying ω_i in Eq. (11), the equations of motion, Eqs. (8–10), upon neglecting all except linear starred terms, take the form

$$(d\omega_1^*/dt) = 0 \tag{12}$$

$$(I_2 + I)(d\omega_2^*/dt) + [(I + I_1 - I_3)\omega_0 + 2I\Omega]\omega_3^* = 0 \tag{13}$$

$$(I_3 + I)(d\omega_3^*/dt) + [(-I + I_2 - I_1)\omega_0 - 2I\Omega]\omega_2^* = 0 \tag{14}$$

Eliminating ω_2^* from Eqs. (13) and (14) leads to the expression

$$(I_2 + I)(I_3 + I)(d^2\omega_3^*/dt^2) + [(2I\Omega + I\omega_0)^2 + (I_1 - I_2)(I_1 - I_3)\omega_0^2 + (2I\Omega\omega_0 + I\omega_0^2) \times (2I_1 - I_2 - I_3)]\omega_3^* = 0 \tag{15}$$

It is shown easily that eliminating ω_3^* from Eqs. (13) and (14) leads to an identical equation for ω_2^*. The condition of stability, that is, the condition such that ω_3^* (and hence ω_2^*) remain small clearly is fulfilled by requiring the coefficient of ω_3^* in Eq. (15) to be positive. Furthermore, Eq. (12) shows that ω_1^* will remain small.

Case II: Stability of uniform rotation with oscillating gyro

Consider the case when the motion of G in V is an oscillation such that Ω is given by

$$\Omega = Ap \cos pt \tag{16}$$

where A and p are constants. Let V again be rotating about an axis parallel to $\mathbf{n}_1$.

Following the procedure used in Case I, the angular velocity of V is given by Eq. (11). By identifying ω_i and using Eq. (16), the first equation of motion, Eq. (8), upon neglecting all except linear starred terms, takes the form

$$(I_1 + 2I)(d\omega_1^*/dt) - 2IAp^2 \sin pt \tag{17}$$

From this equation it is clear that IAp^2 must be of the order of magnitude of the starred terms to insure stability. Noting this, and eliminating ω_2^* from the two remaining equations of motion leads to the equation

$$(I_2 + I)(I_3 + I)(d^2\omega_3^*/dt^2) + \{\omega_0^2[(I_1 - I_3)(I_1 - I_2) + I(2I_1 - I_2 - I_3) + I^2] + 2\omega_0 I(2I + 2I_1 - I_2 - I_3)Ap \cos pt\}\omega_3^* = 0 \qquad (18)$$

which may be written easily in the form

$$(d^2\omega_3^*/d\tau^2) + (\delta + \epsilon \cos\tau)\omega_3^* = 0 \qquad (19)$$

Furthermore, it easily is shown that by eliminating ω_3^* from the motion equations leads to an identical equation for ω_2^*.

The conditions for stability of this Mathieu equation are well known and may be found, for example, in Stoker's "Nonlinear Vibrations".[3] Basically, the stability depends upon the values of δ and ϵ. It is known that by proper choice of ϵ, stability may be attained even for negative δ and also instability may be attained for positive δ.

Discussion

In the first case considered, it is seen in Eq. (15) that, if no gyro were present, that is, if I is vanishingly small, there would be stability of rotation only if the axis of rotation is parallel to a maximum or minimum inertia axis of the vehicle for its mass center. This is a well-known result. However, with a rotating gyro within the vehicle, it is clear that stability may be attained irrespective of the inertia properties or rotation of the vehicle by simply making the product $I\Omega$ large enough.

In the second case, it is seen in Eq. (18) or (19) that rotational stability of the vehicle may be attained (or disrupted) simply by oscillation of the gyro, the stability being dependent upon the frequency and amplitude of oscillation.

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Approximations for Supersonic Flow over Cones

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Nomenclature

C_p = surface pressure coefficient
f_1, f_2, f_3 = functions defined by Eqs. (3–5)
g_1, g_2, g_3 = functions defined by Eqs. (6–8)
M = Mach number
γ = ratio of specific heats
δ = cone angle
θ = shock wave angle

Subscripts

∞ = freestream conditions
c = conditions on cone surface
M = conditions at the largest cone angle that permits an attached shock

Received April 15, 1963.

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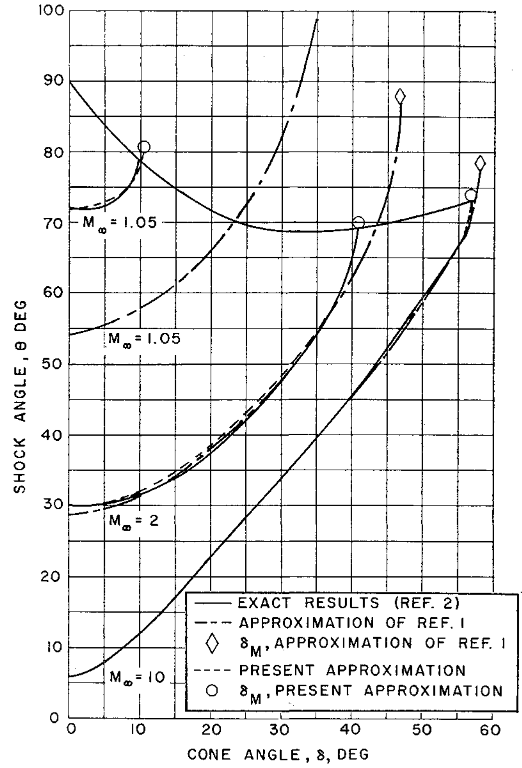

Fig. 1 Comparison of shock angle approximations (γ = 1.405)

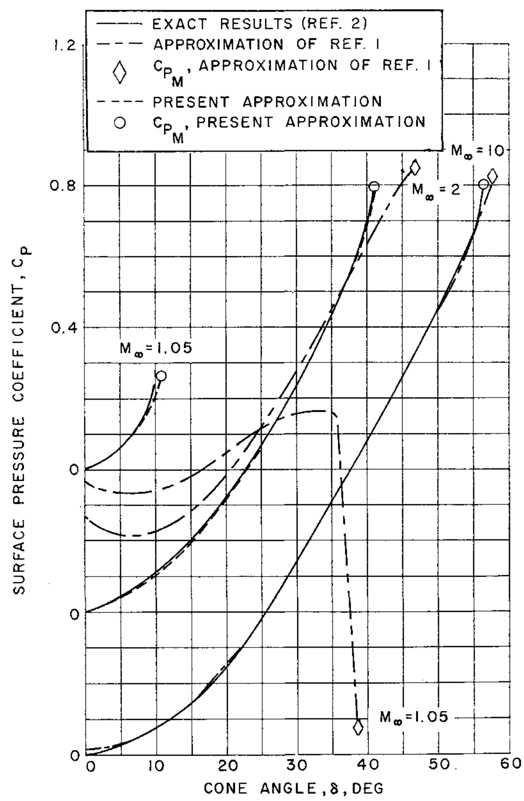

Fig. 2 Comparison of surface pressure coefficient approximations (γ = 1.405)

In many applications, it is necessary to compute the shock angle, surface pressure, and surface Mach number for a cone at zero angle of attack in supersonic flow. The independent variables are freestream Mach number, cone angle, and specific heat ratio of the gas. Generally, three courses of action are possible. First, the differential equation of the flow could be solved for each case. Second, tables of solu-